Spectral Change Point Estimation via Sparse Tensor Decomposition

Based on Xinyu Zhang & Kung-Sik Chan (2024)

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Background

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Spectrum captures dynamic covariance structure across all lags:

- identifies periodic patterns,
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However, two fundamental challenges remain:

- structural changes are often **sparse** in the series dimension;
- changes may occur only within **specific frequency bands**;
- existing methods cannot jointly identify **which series** and **which frequencies** are responsible for each change.

Problem Overview

High-dimensional multivariate time series may undergo **structural changes** in their dependence structure.
The main tasks are to:

- determine **when** the dependence structure changes,
- identify **which series** are affected,
- and pinpoint **at which frequencies** the changes occur.

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Key idea:

- raw covariance captures only lag-0 dependence,
- while the **spectral structure** summarizes dependence across all lags,
- therefore, blockwise spectral density matrices are analyzed instead of raw covariance.

Data Model and Block Structure

We observe a p-dimensional multivariate time series:

$$\mathbf{X} = (\mathbf{X}_1, \dots, \mathbf{X}_N), \qquad \mathbf{X}_t \in \mathbb{R}^p.$$

To study how the dependence structure evolves over time, we divide the series into **blocks**:

$$\text{block } b = \{\mathbf{X}_t : t \in I_b\}, \qquad b = 1, \dots, B.$$

Within each block, the process is assumed to be approximately stationary. This allows us to estimate its **local spectral characteristics**.

This blockwise structure is the foundation for detecting spectral changes across time.

Blockwise Spectral Matrix Estimation

For each block b, its **local spectral density** is estimated using a smoothed periodogram:

$$\hat{f}_b(\omega) = \frac{1}{2\pi} \sum_{m=-R}^{R} K\left(\frac{m}{R}\right) \hat{\Sigma}_b(m)\, e^{-i\omega m}, \qquad \omega \in [0, \pi].$$

Here:

- $K(\cdot)$ is a kernel function (Bartlett kernel in the implementation);
- $\hat{\Sigma}_b(m)$ is the lag-m sample autocovariance within block b.

Interpretation:

- Each entry $\hat{f}_b(\omega)_{ij}$ measures the association between series i and j at frequency ω;
- Different blocks correspond to different time periods;
- Changes in $\hat{f}_b(\omega)$ across b indicate **structural breaks** in the system.

From Spectral Matrices to a Tensor Structure

For each block b and frequency ω, we have a $p \times p$ spectral density matrix $\hat{f}_b(\omega)$.

Collecting these matrices across blocks forms a **3-way tensor representation**:

$$T(:,:,b) = \hat{f}_b(\omega), \qquad T \in \mathbb{R}^{p \times p \times B}.$$

Interpretation of tensor modes:

- Mode-1: series index i,
- Mode-2: series index j,
- Mode-3: block (time) index b.

This tensor encodes how the **frequency-domain dependence** between all pairs of series evolves over time.

Tensor CUSUM for Spectral Change-Point Detection

To detect structural breaks, spectral matrices from different time blocks are compared.

A **CUSUM tensor** is constructed as a discrepancy measure between the first b blocks and the remaining $B - b$ blocks:

$$\mathcal{C}(b) = \sqrt{\frac{B-b}{bB}} \sum_{t \leq b} T(:,:,t) \; - \; \sqrt{\frac{b}{(B-b)B}} \sum_{t > b} T(:,:,t).$$

Interpretation:

- large values of $\mathcal{C}(b)$ indicate a strong discrepancy between the two segments;
- change points correspond to positions where this discrepancy attains a peak;
- each $\mathcal{C}(b)$ is a $p \times p$ matrix, revealing which pairs of series contribute to the change.

Low-Rank Projection: Extracting Informative Directions

The CUSUM matrix $\mathcal{C}(b)$ is $p \times p$, and its estimation can be unstable in high-dimensional settings due to noise accumulation.

To strengthen the underlying change signal, a **low-rank projection** is considered by solving

$$\max_{\|u\|=\|v\|=1} \; u^\top \mathcal{C}(b)\, v,$$

which searches for the most informative rank-one contrast.

Why low-rank?

- structural changes typically manifest in only a few dominant directions;
- projecting onto a low-rank subspace reduces high-dimensional noise;
- the resulting vectors (u, v) indicate which variables contribute most to the structural change.

Frequency-Specific Projection Approach

For each frequency ω, the CUSUM tensor $T_{1,B}(\omega)$ contains information about structural changes.

Goal:

- find a projection vector $\beta(\omega) \in \mathbb{R}^p$
- such that the projected series preserves the largest change signal.

Projection:

$$F(\omega) \times_1 \beta(\omega) \times_2 \beta(\omega)$$

turns the $p \times p \times B$ tensor into a univariate series.

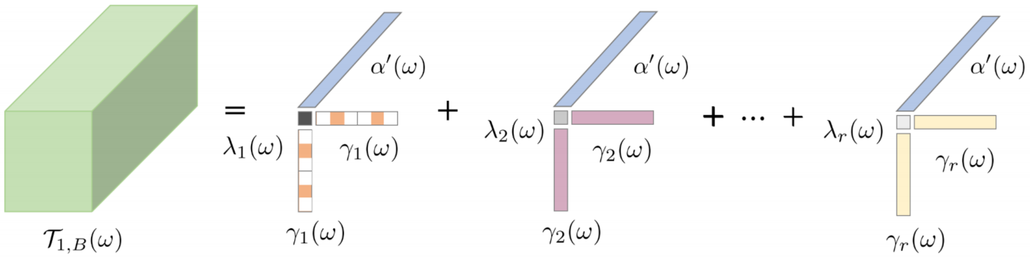
The optimal projection is the **leading mode-1 component** in the tensor decomposition of $T_{1,B}(\omega)$.

Special Tensor Structure

The CUSUM tensor $T_{1,B}(\omega)$ has the form $T_{1,B}(\omega) = g(\omega) \circ \alpha'(\omega)$.
Let the eigendecomposition of $g(\omega)$ be $g(\omega) = \sum_{i=1}^{r} \lambda_i(\omega)\, \gamma_i(\omega) \circ \gamma_i(\omega)$.
Normalizing $\alpha'(\omega)$ as $\alpha(\omega) = \frac{\alpha'(\omega)}{\|\alpha'(\omega)\|}$, obtaining the CP form:

$$T_{1,B}(\omega) = \sum_{i=1}^{r} \lambda_i(\omega)\, \|\alpha'(\omega)\|\, \gamma_i(\omega) \circ \gamma_i(\omega) \circ \alpha(\omega).$$

This structure enables efficient extraction of the leading component $\gamma_1(\omega)$.



Extracting the Leading Component

Given the CP structure

$$T_{1,B}(\omega) = \sum_{i=1}^{r} \lambda_i(\omega) \, \|\alpha'(\omega)\| \, \gamma_i(\omega) \circ \gamma_i(\omega) \circ \alpha(\omega),$$

the goal is to recover the leading spatial component $\gamma_1(\omega)$.

Key observations:

- mode-3 is shared across all rank-1 terms,
- modes 1 and 2 are identical and symmetric,
- the tensor behaves like a "rank-1" object once projected along mode-3.

Projection along mode-3:

$$T_{1,B}(\omega) \times_3 \alpha(\omega) \;\propto\; \sum_{i=1}^{r} \lambda_i(\omega) \, \gamma_i(\omega) \circ \gamma_i(\omega).$$

This reduces the problem to finding the leading eigenvector of a $p \times p$ symmetric matrix.

Extraction of the Leading Component

- **Key idea:** Use tensor contraction + sparse matrix power to recover the dominant spatial component $\gamma_1(\omega)$.

Algorithm 1: Component Extraction

Input: CUSUM tensor $T_{1,B}(\omega)$, matrix $g(\omega)$, sparsity s, iterations T
Output: Leading component $\gamma_1(\omega)$

Initialize: Random unit-norm vector $\gamma^{(0)}(\omega)$

for $t = 1, 2, \ldots, T$ **do**

$v \leftarrow T_{1,B}(\omega) \times_1 \gamma^{(t-1)}(\omega) \times_2 \gamma^{(t-1)}(\omega)$
$v \leftarrow g(\omega)\, v$
$\gamma^{(t)}(\omega) \leftarrow \text{Truncate}(v, s)$
$\gamma^{(t)}(\omega) \leftarrow \gamma^{(t)}(\omega) / \|\gamma^{(t)}(\omega)\|_2$

end for

Full Change-Point Detection Algorithm

Algorithm 3: Multiple spectral change points detection algorithm

Input: Time series $X_t \in \mathbb{R}^p$, block length B, sparsity s, iterations T.
Output: Estimated change points.

1. **Spectral estimation:** Compute spectral density estimates $\hat{f}_b(\omega)$ for each block b.
2. **CUSUM tensor construction:** Form the tensor $T_{1,B}(\omega) = g(\omega) \circ \alpha'(\omega)$.
3. **Leading component estimation:** For each frequency ω, compute $\gamma_1(\omega)$ using Algorithm 1.
4. **Projected CUSUM statistic:** Compute $S_b(\omega) = \gamma_1(\omega)^\top \hat{f}_b(\omega)\gamma_1(\omega)$.
5. **Aggregate over frequencies:** $S_b = \sum_\omega S_b(\omega)$.
6. **Change-point detection:** Identify peaks of S_b over $b = 1, \dots, B$.

Simulation Setup

Goal: Evaluate spectral change-point detection under different high-dimensional time series models.

Single Change Point (DGP1)

- Model: High-dimensional factor model $X_t = \Lambda F_t + \varepsilon_t, \quad t = 1, \dots, N.$
- Purpose: Assess performance under sparse and weak spectral shifts.

Multiple Change Points (DGP2, DGP3)

- **DGP2:** VMA(2) model $X_t = \sum_{k=0}^{2} B_{q,k}\, \varepsilon_{t-k}, \quad t \in (\tau_{q-1}, \tau_q].$
- **DGP3:** VAR(2) model $X_t = A_{q,1} X_{t-1} + A_{q,2} X_{t-2} + \varepsilon_t.$
- Purpose: Evaluate robustness in complex multi-change-point scenarios.

Simulation Data Results

Table 2: Simulation results for DGP2 with $N = 6000$

N	k_0	$\hat{q} < q$	$\hat{q} = q$	$\hat{q} > q$	ARI	$\hat{q} < q$	$\hat{q} = q$	$\hat{q} > q$	ARI	$\hat{q} < q$	$\hat{q} = q$	$\hat{q} > q$	ARI
		SCP				SBS-LSW				Series-by-series			
6000	3	0.02	0.98	0.00	0.98	0	0.82	0.18	0.98	0	0.19	0.81	0.90
	8	0.01	0.99	0.00	0.99	0	0.76	0.24	0.98	0	0.21	0.79	0.90
	40	0.00	1.00	0.00	1.00	0	0.31	0.69	0.95	0	0.42	0.58	0.91
	80	0.00	1.00	0.00	1.00	0	0.11	0.89	0.92	0	0.92	0.08	0.95
		FVAR-c				FVAR-i				FreSpeD			
	3	1.00	0.00	0.00	0.00	0	1.00	0.00	0.98	0	0.08	0.92	0.89
	8	0.93	0.07	0.00	0.33	0	1.00	0.00	0.98	0	0.01	0.99	0.80
	40	0.00	0.92	0.08	0.97	0	1.00	0.00	0.98	0	0.00	1.00	0.64
	80	0.00	0.72	0.28	0.95	0	1.00	0.00	0.98	0	0.00	1.00	0.66

Real Data Experiment Setup

Dataset: S&P 100 Constituent Returns

- Daily log-returns of S&P100 constituent stocks.
- Time period: 2000–2021.
- Dimension: $p = 79$ stocks (after filtering missing series).
- Sample size: $N = 5520$ trading days.

Preprocessing

- Returns standardized to zero mean and unit variance.
- Heavy-tailed behavior addressed using the normal quantile transform.
- Spectral density estimated over blocks of size $L = 60$ (approx. 3 months per block).

Real Data Results

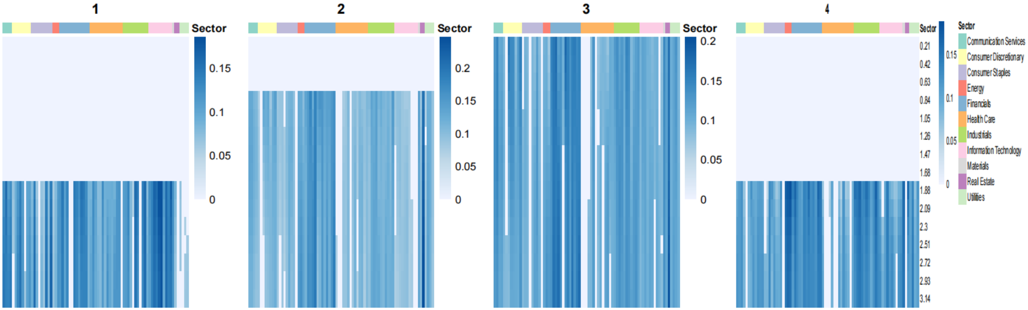


Figure 6.1: Projection per frequency (each row) and series (each column) for the four change points.

Thank you!