Tensor Completion via Collaborative Sparse and Low-Rank Transforms

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Outline

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Third-Order Tensor Data

Many real-world data are third-order tensors:

- color images
- videos
- multispectral/hyperspectral images
- traffic/internet data
- $\cdots$

Missing Values

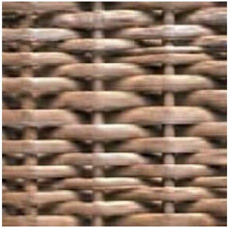
Multi-dimensional data usually undergo missing entries or undersample problem due to sensor malfunction or poor atmospheric conditions, which hinders its subsequent applications.

Tensor completion

Tensor completion refers to the process of inferring missing values from partially observed tensor data.

Low-rankness

Fortunately, real-world data are not unstructured. In the matrix case, the rank is a powerful tool to capture global information.



For observed matrix $\mathbf{O} \in \mathbb{R}^{n_1 \times n_2}$, the low-rank matrix completion is mathematically formulated as follows:

$$\begin{aligned} \min_{\mathbf{X}} \ & \operatorname{rank}(\mathbf{X}) \\ \text{s.t.} \ & \mathbf{X}_\Omega = \mathbf{O}_\Omega, \end{aligned} \tag{1}$$

where $\mathbf{X}$ is the required matrix and Ω is the index set of the observed elements.

For observed tensor $\mathcal{O} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$, the low-rank tensor completion (LRTC) is mathematically formulated as follows:

$$\begin{aligned} \min_{\mathcal{X}} \ & \operatorname{rank}(\mathcal{X}) \\ \text{s.t.} \ & \mathcal{X}_\Omega = \mathcal{O}_\Omega. \end{aligned} \tag{2}$$

The Rank of Tensors

However, the definition of the rank of tensors is the fundamental problem, which is still an open problem:

- CP rank
- Tucker rank
- tubal rank or multi rank
- tensor train rank
- tensor ring rank
- $\cdots$

Outline

Tensor singular values decomposition (t-SVD)

The tensor-tensor-product (t-prod) $\mathcal{C} = \mathcal{A} * \mathcal{B}$ of $\mathcal{A} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$ and $\mathcal{B} \in \mathbb{R}^{n_2 \times n_4 \times n_3}$ is a tensor of size $n_1 \times n_4 \times n_3$, where the (i,j)-th tube $\mathbf{c}_{ij:}$ is given by

$$\mathbf{c}_{ij:} = \mathcal{C}(i,j,:) = \sum_{k=1}^{n_2} \mathcal{A}(i,k,:) * \mathcal{B}(k,j,:)$$

where $*$ denotes the circular convolution between two tubes of same size.

M. E. Kilmer and C. D. Martin, Factorization Strategies for Third-order Tensors, ***Linear Algebra and its Applications***, 2011

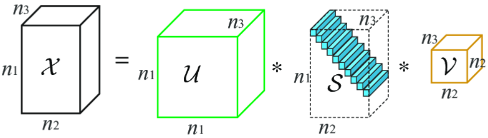
Tensor Singular Value Decomposition (t-SVD)

Based on the tensor-tensor product, the tensor singular value decomposition has been emerged as a powerful tool for multi-dimensional image processing:

$$\mathcal{X} = \mathcal{U} * \mathcal{S} * \mathcal{V}^H,$$

where $\mathcal{U}$ and $\mathcal{V}$ are orthogonal tensors, $\mathcal{S}$ is the f-diagonal tensor, and $\mathcal{V}^H$ denotes the conjugate transpose of $\mathcal{V}$.

Based on t-SVD, the tubal rank of $\mathcal{X}$ is defined as the number of non-zero tubes of $\mathcal{S}$.



M. E. Kilmer and C. D. Martin, Factorization Strategies for Third-order Tensors, ***Linear Algebra and its Applications***, 2011

Tensor Nuclear Norm (TNN)

Based on tubal rank, the tensor nuclear norm (TNN) was suggested as the convex surrogate to capture the intrinsic structure of the underlying tensor. For $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$, TNN is defined as

$$\|\boldsymbol{\mathcal{X}}\|_{\mathrm{TNN}} := \sum_{i=1}^{r} \mathcal{S}(i,i,1),$$

where $\mathcal{S}(i,i,1), i = 1, \cdots, r$ are singular values of $\mathcal{X}$.

Z. M. Zhang, et al., Novel Methods for Multilinear Data Completion and De-noising Based on Tensor-SVD, ***CVPR***, 2014

C. Y. Lu, et al., Tensor Robust Principal Component Analysis with A New Tensor Nuclear Norm, ***IEEE TPAMI***, 2020

TNN-based LRTC

Therefore, the TNN-based LRTC problem can be rewritten as follows:

$$\begin{aligned} &\min_{\mathcal{X}} \|\mathcal{X}\|_{\text{TNN}} \\ &\text{s.t. } \mathcal{X}_{\Omega} = \mathcal{O}_{\Omega} \end{aligned} \tag{3}$$

Outline

The Revisit of Tensor-Tensor Product

For $\mathcal{A} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$ and $\mathcal{B} \in \mathbb{R}^{n_2 \times n_4 \times n_3}$, the tensor-tensor product is equivalent to

$$\overline{\mathcal{C}}(:,:,k) = \overline{\mathcal{A}}(:,:,k)\overline{\mathcal{B}}(:,:,k), k = 1, \cdots, n_3,$$

where $\overline{\mathcal{C}} = \mathcal{C} \times_3 \mathbf{F}$, $\overline{\mathcal{A}} = \mathcal{A} \times_3 \mathbf{F}$, $\overline{\mathcal{B}} = \mathcal{B} \times_3 \mathbf{F}$. Here $\mathbf{F}$ is the discrete Fourier transform matrix and $\times_3$ is the mode-3 product.

Z. M. Zhang, et al., Novel Methods for Multilinear Data Completion and De-noising Based on Tensor-SVD, ***CVPR***, 2014

C. Y. Lu, et al., Tensor Robust Principal Component Analysis with A New Tensor Nuclear Norm, ***IEEE TPAMI***, 2020

The Revisit of Tensor Nuclear Norm

For $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$, tensor nuclear norm is equivalent to

$$\|\mathcal{X}\|_{\mathrm{TNN}} = \sum_{k=1}^{n_3} \|\mathcal{Z}(:,:,k)\|_*,$$

where $\mathcal{Z} = \mathcal{X} \times_3 \mathbf{F}$ and $\mathcal{X} = \mathcal{Z} \times_3 \mathbf{F}^H$. Under the multi-linear algebra framework, the vital important building block is the transform, which captures the relationship between slices.

Z. M. Zhang, et al., Novel Methods for Multilinear Data Completion and De-noising Based on Tensor-SVD, ***CVPR***, 2014

C. Y. Lu, et al., Tensor Robust Principal Component Analysis with A New Tensor Nuclear Norm, ***IEEE TPAMI***, 2020

The TNN-based LRTC

Therefore, the TNN-based LRTC problem can be mathematically formulated as follows:

$$\begin{aligned} &\min_{\mathcal{Z}} \sum_{i=1}^{n_3} \|\mathcal{Z}(:,:,i)\|_* \\ &\text{s.t.} \ \ (\mathcal{Z} \times_3 \mathbf{F}_{n_3}^{-1})_\Omega = \mathcal{O}_\Omega, \end{aligned} \tag{4}$$

where $\mathbf{F}_{n_3}^{-1}$ is the inverse DFT matrix, which is pre-defined and unitary.

DFT	***Unitary***	***Invertible***	***Non-invertible***
Kilmer *et al.* [2011]	Xu *et al.* [2019]	Kernfled *et al.* [2015]	Jiang *et al.* [2020]
	Song et al. [2020]	Lu *et al.* [2019]	**Kong et al. [2021]**
			Jiang et al. [2021]

M. E. Kilmer and C. D. Martin, Factorization Strategies for Third-order Tensors, ***LAA***, 2011

W.-H. Xu, X.-L. Zhao, and M. Ng, A fast algorithm for cosine transform based tensor singular value decomposition, arXiv:1902.03070, 2019.

G. J. Song, et al., Robust Tensor Completion Using Transformed Tensor Singular Value Decomposition, ***NLAA***, 2020

E. Kernfeld, et al., Tensor-tensor Products with Invertible Linear Transforms, ***LAA***, 2015

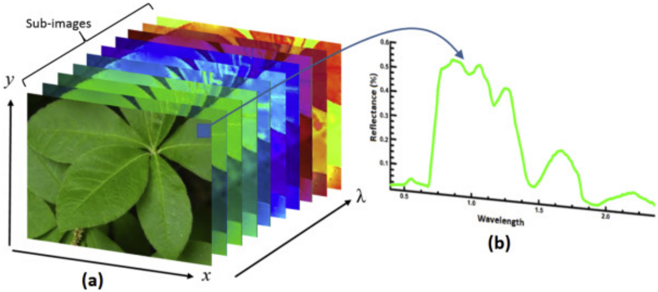
C. Y. Lu, et al., Low-Rank Tensor Completion with a New Tensor Nuclear Norm Induced by Invertible Linear Transforms, ***CVPR***, 2019

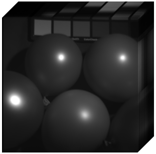
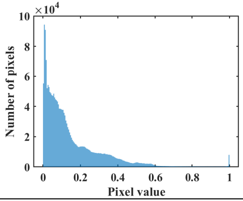
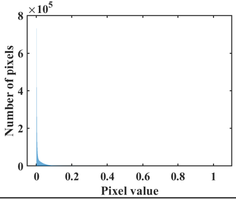
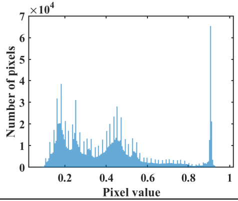
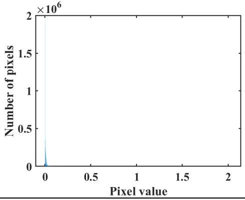
T. X. Jiang, et al., Framelet Representation of Tensor Nuclear Norm for Third-order Tensor Completion, ***IEEE TIP***, 2020

T. X. Jiang, et al., Dictionary Learning with Low-rank Coding Coefficients for Tensor Completion, ***IEEE TNNLS***, 2021, doi:10.1109/TNNLS.2021.3104837.

H. Kong and Z. C. Lin, Tensor Q-Rank: a New Data Dependent Tensor Rank, ***Machine Learning***, 2021

The slice view and fiber view



<table>
<tr><th>Data</th><th>The original tensors</th><th>The transformed tensors</th></tr>
<tr><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td></td></tr>
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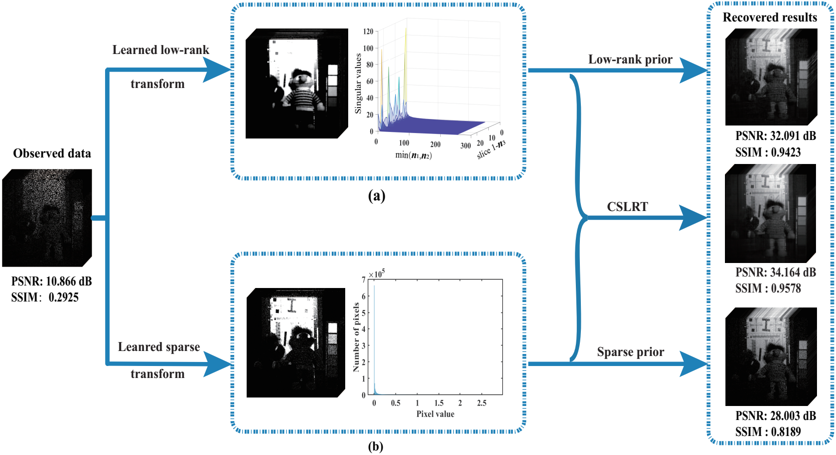

Observed data
PSNR: 10.866 dB
SSIM: 0.2925
Learned low-rank
transform
Singular values
min(n_1,n_2)
slice 1-n_3
(a)
Leanred sparse
transform
$\times 10^5$
Number of pixels
Pixel value
(b)
Low-rank prior
CSLRT
Sparse prior
Recovered results
PSNR: 32.091 dB
SSIM : 0.9423
PSNR: 34.164 dB
SSIM : 0.9578
PSNR: 28.003 dB
SSIM : 0.8189

We suggest the collaborative sparse and low-rank transforms (CSLRT) as follows:

$$\begin{aligned} \min_{\mathcal{X},\mathcal{Z},\mathcal{S},\mathbf{D},\mathbf{Q}} & \sum_{k=1}^{d} \|\mathcal{Z}(:,:,k)\|_* + \lambda\|\mathcal{S}\|_1 \\ \text{s.t.} \;\; & \mathcal{X}_\Omega = \mathcal{O}_\Omega, \mathcal{X} = \mathcal{Z} \times_3 \mathbf{D}, \mathcal{X} = \mathcal{S} \times_3 \mathbf{Q}, \\ & \|\mathbf{D}(:,k)\|_2 = 1 \text{ for } k = 1, ..., d, \\ & \|\mathbf{Q}(:,k)\|_2 = 1 \text{ for } k = 1, ..., q, \end{aligned} \tag{5}$$

where $\|\mathcal{S}\|_1 = \sum_{ijk} |s_{ijk}| = \sum_{i,j} \|\mathcal{S}(i,j,:)\|_1$ is the sum of ℓ_1-norms of all third mode fibers under the transform **Q**. $\mathbf{D} \in \mathbb{R}^{n_3 \times d}$ and $\mathbf{Q} \in \mathbb{R}^{n_3 \times q}$ are the low-rank transform and the sparse transform, and $\mathcal{Z} \in \mathbb{R}^{n_1 \times n_2 \times d}$ and $\mathcal{S} \in \mathbb{R}^{n_1 \times n_2 \times q}$ are tensors under transforms **D** and **Q**, respectively.

Unconstrained Optimization Problem

By defining

$$\Phi(\mathcal{X}) = \begin{cases} 0, & \mathcal{X}_\Omega = \mathcal{O}_\Omega, \\ \infty, & \text{otherwise}, \end{cases}$$

and

$$\Psi(\mathbf{D}) = \begin{cases} 0, & \|\mathbf{D}(:,k)\|_2 = 1 \text{ for } k = 1, ..., d, \\ \infty, & \text{otherwise}, \end{cases}$$

the problem (5) can be rewritten as the following unconstraint problem via half quadratic splitting technique:

PAM algorithm

$$\min_{\mathcal{X},\mathcal{Z},\mathcal{S},\mathbf{D},\mathbf{Q}} \sum_{k=1}^{d} \|\mathcal{Z}(:,:,k)\|_* + \lambda\|\mathcal{S}\|_1 + \frac{\beta_1}{2}\|\mathcal{X} - \mathcal{Z} \times_3 \mathbf{D}\|_F^2 + \frac{\beta_2}{2}\|\mathcal{X} - \mathcal{S} \times_3 \mathbf{Q}\|_F^2 + \Psi(\mathbf{D}) + \Psi(\mathbf{Q}) + \Phi(\mathcal{X}). \tag{6}$$

We denote the objective function in (6) as $L(\mathcal{Z}, \mathbf{D}, \mathcal{S}, \mathbf{Q}, \mathcal{X})$. Under the proximal alternating minimization (PAM) algorithm framework, we can alternatively update each variable.

Update $\mathcal{Z}$ and D

$\mathbf{Z}_{(3)} = \textbf{unfold}_3(\mathcal{Z})$ and $\mathbf{D}$ can be rewritten as follows:

$$\begin{aligned}\mathbf{Z}_{(3)} &= [\mathbf{z}_1, \cdots, \mathbf{z}_k, \cdots, \mathbf{z}_d]^\top \\ &= [\textbf{vec}(\mathbf{Z}_1), \cdots, \textbf{vec}(\mathbf{Z}_k), \cdots, \textbf{vec}(\mathbf{Z}_d)]^\top\end{aligned}$$

and

$$\mathbf{D} = [\mathbf{d}_1, \cdots, \mathbf{d}_k, \cdots, \mathbf{d}_d],$$

where $\mathbf{z}_k = \textbf{vec}(\mathbf{Z}_k)$ denotes the vectorization of $\mathbf{Z}_k$ [1] and $\mathbf{d}_k = \mathbf{D}(:, k)$ is the k-th column of **D**.

[1] For convenience, we denote $\mathbf{Z}_k = \mathcal{Z}(:, :, k)$ as the k-th frontal slice of $\mathcal{Z}$.

Updating $\mathcal{Z}$ and D

Then, $\mathcal{Z}$ and **D** subproblems can be rewritten as follows:

$$\min_{\mathcal{Z}} \sum_{k=1}^{d} \|\mathbf{Z}_k\|_* + \frac{\beta_1^t}{2}\|\mathbf{X}_{(3)}^t - \sum_{k=1}^{d} \mathbf{d}_k \mathbf{z}_k^\top\|_F^2 + \frac{\rho_1}{2}\sum_{k=1}^{d} \|\mathbf{Z}_k - \mathbf{Z}_k^t\|_F^2 \quad (7)$$

and

$$\min_{\mathbf{D}} \frac{\beta_1^t}{2}\|\mathbf{X}_{(3)}^t - \sum_{k=1}^{d} \mathbf{d}_k \mathbf{z}_k^\top\|_F^2 + \frac{\rho_2}{2}\sum_{k=1}^{d} \|\mathbf{d}_k - \mathbf{d}_k^t\|_F^2 + \Psi(\mathbf{D}). \quad (8)$$

Updating $\mathcal{Z}$ and D

The problem (7) and (8) can be solved by solving a sequence of $\mathbf{Z}_k$ and $\mathbf{d}_k$ subproblems, respectively. We define

$$\begin{cases} \widehat{\mathbf{Z}}_k^t = \left[\mathbf{z}_1^{t+1}, \cdots, \mathbf{z}_{k-1}^{t+1}, \mathbf{z}_{k+1}^t, \cdots, \mathbf{z}_d^t\right]^\top, \\ \widehat{\mathbf{D}}_k^t = [\mathbf{d}_1^{t+1}, \cdots, \mathbf{d}_{k-1}^{t+1}, \mathbf{d}_{k+1}^t, \cdots, \mathbf{d}_d^t], \\ \mathbf{R}_k^t = \mathbf{X}_{(3)}^t - \widehat{\mathbf{D}}_k^t \widehat{\mathbf{Z}}_k^t. \end{cases}$$

Then, the $\mathbf{Z}_k$ and $\mathbf{d}_k$ ($k = 1, ..., d$) are updated as follows:

$$\min_{\mathbf{Z}_k} \frac{\beta_1^t}{2} \|\mathbf{R}_k^t - \mathbf{d}_k^t \mathbf{z}_k{}^\top\|_F^2 + \|\mathbf{Z}_k\|_* + \frac{\rho_1}{2} \|\mathbf{Z}_k - \mathbf{Z}_k^t\|_F^2 \tag{9}$$

and

$$\min_{\mathbf{d}_k} \frac{\beta_1^t}{2} \|\mathbf{R}_k^t - \mathbf{d}_k \mathbf{z}_k^{t+1\,\top}\|_F^2 + \frac{\rho_2}{2} \|\mathbf{d}_k - \mathbf{d}_k^t\|_F^2 + \Psi(\mathbf{D}). \tag{10}$$

Updating $\mathcal{Z}$ and D

After combining two quadratic terms in problem (9), we can directly derive the closed-form solution of problem (9) via the singular value thresholding (SVT) operator as follows:

$$\mathbf{Z}_k^{t+1} = \mathcal{T}_{\frac{1}{\beta_1^t+\rho_1}}\left(\frac{\beta^t \mathrm{vec}^{-1}(\mathbf{R}_k^{t\top}\mathbf{d}_k^t) + \rho_1 \mathbf{Z}_k^t}{\beta_1^t + \rho_1}\right).$$

Similarly, we can obtain the closed-form solution of (10) as follows:

$$\mathbf{d}_k^{t+1} = \frac{\beta_1^t \mathbf{R}_k^t \mathrm{vec}(\mathbf{Z}_k^{t+1}) + \rho_2 \mathbf{d}_k^t}{\|\beta_1^t \mathbf{R}_k^t \mathrm{vec}(\mathbf{Z}_k^{t+1}) + \rho_2 \mathbf{d}_k^t\|_2}.$$

Updating $\mathcal{S}$ and Q

$$\mathbf{S}_k^{t+1} = \texttt{soft}_{\frac{1}{\beta_2^t+\rho_3}}\left(\frac{\beta_2^t \texttt{vec}^{-1}({\mathbf{R}_k^t}^\top \mathbf{q}_k^t) + \rho_3 \mathbf{S}_k^t}{\beta_1^t + \rho_3}\right).$$

$$\mathbf{q}_k^{t+1} = \frac{\beta_2^t \mathbf{R}_k^t \texttt{vec}(\mathbf{S}_k^{t+1}) + \rho_4 \mathbf{q}_k^t}{\|\beta_2^t \mathbf{R}_k^t \texttt{vec}(\mathbf{S}_k^{t+1}) + \rho_4 \mathbf{q}_k^t\|_2}.$$

Updating $\mathcal{X}$

We update $\mathcal{X}$ by solving the following minimization problem:

$$\mathcal{X}^{t+1} = \underset{\mathcal{X}}{\text{argmin}} \frac{\beta_1^t}{2}\|\mathcal{X} - \mathcal{Z}^{t+1} \times_3 \mathbf{D}^{t+1}\|_F^2 + \frac{\beta_2^t}{2}\|\mathcal{X} - \mathcal{S}^{t+1} \times_3 \mathbf{Q}^{t+1}\|_F^2 + \frac{\rho_5}{2}\|\mathcal{X} - \mathcal{X}^t\|_F^2 + \Phi(\mathcal{X}).$$

Therefore, $\mathcal{X}^{t+1}$ is updated via the following steps:

$$\begin{cases} \mathcal{X}^{t+\frac{1}{2}} = \dfrac{\beta_1^t \mathcal{Z}^{t+1} \times_3 \mathbf{D}^{t+1} + \beta_2^t \mathcal{S}^{t+1} \times_3 \mathbf{Q}^{t+1} + \rho_5 \mathcal{X}^t}{\rho_5 + \beta_1^t + \beta_2^t}, \\ \mathcal{X}^{t+1} = (\mathcal{X}^{t+\frac{1}{2}})_{\Omega^C} + \mathcal{O}_{\Omega}, \end{cases} \tag{11}$$

where Ω^C denotes the complementary set of Ω.

Theoretical guarantee

Under the PAM framework, we have the following theoretical guarantee:

Theorem

The sequence $\{\mathcal{Z}^t, \mathbf{D}^t, \mathcal{S}^t, \mathbf{Q}^t, \mathcal{X}^t\}_{t\in\mathbb{N}}$ *generated by PAM-based Algorithm is bounded and converges to a critical point of L.*

3DCSLRT

Moreover, for tensors with limited correlation along the third mode (e.g., color images), we further suggest three-dimensional CSLRT (3DCSLRT) as follows:

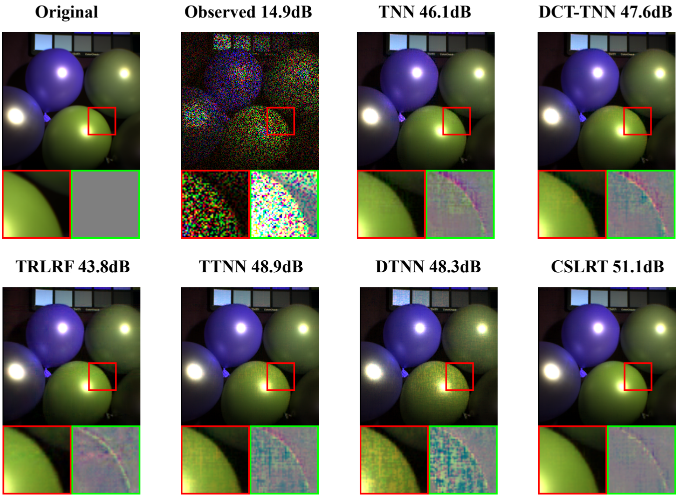
$$\min_{\mathcal{X},\mathcal{Z}_i,\mathcal{S}_i,\mathbf{D}_i,\mathbf{Q}_i} \sum_{i=1}^{3} \alpha_i \Big(\| \mathrm{bdiag}(\mathcal{Z}_i) \|_* + \lambda_i \|\mathcal{S}_i\|_1 \Big)$$

$$\text{s.t.} \;\; \mathcal{X}_\Omega = \mathcal{O}_\Omega, \mathcal{X} = \mathcal{Z}_i \times_i \mathbf{D}_i, \mathcal{X} = \mathcal{S}_i \times_i \mathbf{Q}_i, \quad (12)$$

$$\|\mathbf{D}_i(:,k)\|_2 = 1 \text{ for } k = 1, ..., d_i,$$

$$\|\mathbf{Q}_i(:,k)\|_2 = 1 \text{ for } k = 1, ..., q_i,$$

Numerical experiments



Ben-Zheng Li, Xi-Le Zhao*, Jian-Li Wang, Yong Chen, Tai-Xiang Jiang, and Jun Liu, Tensor Completion via Collaborative Sparse and Low-Rank Transforms, ***IEEE Transactions on Computational Imaging (TCI)***, Accept with mandatory minor revisions.

Thank you very much for listening.

Wechat